OCR Further Pure 2 Module Revision Sheet

The FP2 exam is 1 hour 30 minutes long. You are allowed a graphics calculator.

Before you go into the exam make sure you are fully aware of the contents of the formula booklet you receive. Also be sure not to panic; it is not uncommon to get stuck on a question (I've been there!). Just continue with what you can do and return at the end to the question(s) you have found hard. If you have time check all your work, especially the first question you attempted. . . always an area prone to error.

J.M.S.

Rational Functions

- Review all partial fractions and polynomial division work from C4 before starting this section.
- A nice 'trick' that can be used from time-to-time is to make the top line of an algebraic fraction look like a multiple of the bottom. Then you can split it up. For example

$$\frac{x-1}{x+3} = \frac{x+3-4}{x+3} = 1 - \frac{4}{x+3},$$
$$\frac{2x^2-1}{x-2} = \frac{2x^2-4x+4x-1}{x-2} = \frac{2x(x-2)+4x-1}{x-2} = 2x + \frac{4x-8+7}{x-2} = 2x+4+\frac{7}{x-2}.$$

  Some students like this & others don't; it's up to you if you use it. You could just use polynomial division.
- For some reason best known to the examiners at OCR, C4 only contains two of the three partial fraction types[1]. In C4 you dealt with $\frac{ax+b}{(cx+d)(ex+f)}$ and $\frac{ax^2+bx+c}{(dx+e)(fx+g)^2}$. In FP2 you also need to know how do deal with $\frac{ax^2+bx+c}{(dx+e)(fx^2+g)}$. The general technique is

$$\frac{ax^2+bx+c}{(dx+e)(fx^2+g)} \equiv \frac{A}{dx+e} + \frac{Bx+C}{fx^2+g}.$$

  Remember that to use 'pure' partial fractions the numerator has to have order less than the denominator.
- For example to express $\frac{5x^2-7x+14}{(x-3)(2x^2+1)}$ in partial fractions we start:

$$\frac{5x^2-7x+14}{(x-3)(2x^2+1)} \equiv \frac{A}{x-3} + \frac{Bx+C}{2x^2+1},$$
$$\Rightarrow \qquad 5x^2-7x+14 \equiv (2x^2+1)A + (x-3)(Bx+C).$$

  Clearly a good x-value to use is $x=3$, so

$$x=3 \qquad \Rightarrow \qquad 45-21+14 \equiv 19A \qquad \Rightarrow \qquad \underline{A=2}.$$

  We've no more cunning values so just use $x=0$ to find $14 = 2-3C$, which gives $\underline{C=-4}$.

[1]Note that the wonderful MEI has all three in C4 which is much more coherent. . .

Next use $x = 1$ to give $12 = 6 + (-2)(B - 4)$, which solves to $\underline{B = 1}$. Therefore

$$\frac{5x^2 - 7x + 14}{(x-3)(2x^2+1)} \equiv \frac{2}{x-3} + \frac{x-4}{2x^2+1}.$$

- You can always make the leap from polynomial division to partial fractions in one go if you like. For example to divide

$$\frac{3x^4 + 13x^3 + 27x^2 + 56x + 59}{x^3 + 3x^2 + 4x + 12}$$

we have '$\frac{\text{quartic}}{\text{cubic}}$'. We are therefore expecting 'linear + $\frac{\text{quadratic}}{\text{cubic}}$'. But, because the denominator can be factorised to $(x+3)(x^2+4)$, we could split the '$\frac{\text{quadratic}}{\text{cubic}}$' term into partial fractions too. So

$$\frac{3x^4 + 13x^3 + 27x^2 + 56x + 59}{(x+3)(x^2+4)} \equiv Ax + B + \frac{C}{x+3} + \frac{Dx+E}{x^2+4},$$
$$3x^4 + 13x^3 + 27x^2 + 56x + 59 \equiv (Ax+B)(x+3)(x^2+4) + C(x^2+4) + (Dx+E)(x+3),$$

Clearly $A = 3$ by considering the x^4 coefficient. Running through the rest of the calculations in the usual way (you should do this yourself) we find

$$\frac{3x^4 + 13x^3 + 27x^2 + 56x + 59}{(x+3)(x^2+4)} \equiv 3x + 4 + \frac{2}{x+3} + \frac{x+1}{x^2+4}.$$

Graphs

- To sketch a graph of $y = \frac{f(x)}{g(x)}$ there are a series of steps to follow. If one step contradicts another, chances are you've made a mistake. Firstly to find where a curve crosses the x-axis, set $y = 0$ and solve. Similarly to find where a curve crosses the y-axis, set $x = 0$ and solve.

- To find stationary points just solve $\frac{dy}{dx} = 0$ as usual. To discover their nature you can use the second derivative as normal; or the lo-tech approach. Review your C1 notes.

- To find the vertical asymptotes of the curve $y = \frac{f(x)}{g(x)}$ you need to find where $g(x) = 0$. So $y = \frac{x+3}{(2x-1)(x+2)}$ will have vertical asymptotes $x = -2$ and $x = \frac{1}{2}$.

- To find a horizontal asymptotes of $y = \frac{f(x)}{g(x)}$ you must look at the order of $f(x)$ and the order of $g(x)$.

  1. If "order of $f(x)$" < "order of $g(x)$" then, as $x \to \pm\infty$, $g(x)$ is much, *much* larger than $f(x)$, so $y = 0$ is the horizontal asymptote.
  2. If "order of $f(x)$" = "order of $g(x)$" then, as $x \to \pm\infty$, the dominant term of $f(x)$ and $g(x)$ becomes the largest power of x. Therefore the horizontal asymptote becomes the ratio of the coefficients of the largest power of x. For example $y = \frac{7x^2+2x-1}{5x^2-x-2}$ has $y = \frac{7}{5}$ as its horizontal asymptote.

     (Another, possibly better, way of thinking about this to divide the improper fraction into quotient and remainder by polynomial division. Since the order of the numerator is the same as the order of the denominator, then the quotient is a constant. This constant is the value of the horizontal asymptote.)

3. If "order of $f(x)$" $>$ "order of $g(x)$" then there is *no* horizontal asymptote because as $x \to \pm\infty$, $f(x)$ is much, *much* larger than $g(x)$, so y is unbounded, heading up to $+\infty$ or down to $-\infty$ (just think about what happens when x gets really big).

- If the numerator has order one more than the denominator, there will exist an *oblique asymptote*; a line of the form $y = mx + c$ that the curve approaches when $x \to \pm\infty$. To find this line, you must carry out the polynomial division and find the quotient and remainder. For example $y = \frac{3x^2+4x+5}{x+1}$. We know $y = \frac{3x^2+4x+5}{x+1} = Ax + B + \frac{C}{x+1}$ and, carrying out the calculation (do it yourself!), we find $y = 3x + 1 + \frac{4}{x+1}$. Therefore the oblique asymptote is $y = 3x + 1$ because the $\frac{4}{x+1} \to 0$ as $x \to \pm\infty$.

  Similarly you should find (again, do it yourself!) $y = \frac{-x^3+x+2}{x^2+x+1} = -x + 1 + \frac{x+1}{x^2+x+1}$, so the oblique asymptote is $y = -x + 1$.

- You must be able to discover the range of y-values for which the curve exists, and, equivalently, the values for which it does not. This can be done by finding the stationary points on the curve and considering a sketch. However, there is quite a neat algebraic method. For example: Find the values of y for which the curve

$$y = \frac{x^2 + x + 1}{x^2 + 1}$$

  exists. Multiplying by the denominator we discover $y(x^2 + 1) = x^2 + x + 1$. This can be rearranged as a quadratic in x: $(y - 1)x^2 - x + (y - 1) = 0$. For the curve to exist, we need the quadratic to have at least one solution, so $b^2 - 4ac \geqslant 0$. So

$$\begin{aligned} (-1)^2 - 4(y-1)(y-1) &\geqslant 0, \\ 4y^2 - 8y + 3 &\leqslant 0, \\ (2y-3)(2y-1) &\leqslant 0. \end{aligned}$$

  This quadratic inequality solves to $\frac{1}{2} \leqslant y \leqslant \frac{3}{2}$. So the curve only exists between the horizontal lines $y = \frac{1}{2}$ and $y = \frac{3}{2}$.

- Given a graph of $y = f(x)$, you must also be able to sketch the graph of $y^2 = f(x)$. Most students (including myself) mentally re-cast the problem as drawing $y = \pm\sqrt{f(x)}$. Things to look for include

  1. Anything below the x-axis on the original graph (y) is negative and therefore *cannot* be square rooted. Therefore these x-values represent a forbidden region where y^2 doesn't exist.
  2. All y-values above the x-axis get square rooted. Then these new points also get reflected in the x-axis. *Any* graph $y^2 = f(x)$, *must* have a line of symmetry in the x-axis.
  3. All positive y-values on the original graph get square rooted; therefore points on the line $y = 1$ are invariant. If $y > 1$, then they get 'pulled down' towards the x-axis ($\sqrt{100} = 10$). If $y < 1$ then the points get pushed further away from the x-axis ($\sqrt{\frac{1}{4}} = \frac{1}{2}$).
  4. Vertical asymptotes on y remain vertical asymptotes on y^2.
  5. Horizontal asymptotes above the x-axis ($y = k$, say) become horizontal asymptotes $y = \pm\sqrt{k}$.
  6. Any points where the original curve hits the x-axis are also invariant ($\sqrt{0} = 0$). Also, the gradient of any points where y hits the x-axis become vertical on the y^2 graph.
  7. Any stationary point above the x-axis on y ($(2, 16)$, say) remain stationary points at $(2, \pm 4)$, say.

Polar Coordinates

- Polar coordinates are given as points with (r, θ) with the constraints $r \geqslant 0$ and (usually) either $0 \leqslant \theta < 2\pi$ or $-\pi < \theta \leqslant \pi$. The distance from the origin is r and θ is the angle made with the initial line (i.e. the positive x-axis) measured anti-clockwise. The angle constraints are used so that each point in space has a unique angle[2]. The 'pole' is sometimes used to describe the origin of your xy-grid in the context of a polar graph.

- Circles are described by '$r =$ constant'. Lines running out from the pole are described by '$\theta =$ constant'.

- Polar curves are a new way of describing curves by showing a relationship between r and θ. They are usually given in the form $r = f(\theta)$.

- To sketch a polar curve $r = f(\theta)$ there are various tools to help you (in most cases completely analogous to the tools have help you draw $y = f(x)$).

  1. If in doubt, throw θ values into your $r = f(\theta)$ and work out r-values for given θ-values and plot them.
  2. If you are trying to draw $r = f(\theta)$, some students find it helpful to draw $y = f(x)$ to discover the general behavior of f.
  3. Understand that solving $\frac{dr}{d\theta} = 0$ gives you points where the curve is locally closest or furthest from the pole. If $\frac{d^2r}{d\theta^2} > 0$ then it is a point closest to the pole. If $\frac{d^2r}{d\theta^2} < 0$ then it is a point furthest from the pole.
  4. Solving $r = 0$ will give you the θ values (θ_1, say) that represent where the curve drops into the pole. Therefore the line $\theta = \theta_1$ will represent a tangent to the curve.
  5. Look for symmetries in your function. For example $r = \sin\theta$. We know the sine wave has symmetry about $\frac{\pi}{2}$; i.e. $\sin(\pi - \theta) \equiv \sin\theta$. Therefore the line $\theta = \frac{\pi}{2}$ must represent a line of symmetry on the polar curve.

     More generally if you can show that $f(2\alpha - \theta) \equiv f(\theta)$ then $\theta = \alpha$ represents a line of symmetry on the polar curve $r = f(\theta)$.

- To find areas on a polar graph we use the formula

$$\text{Area} = \int_{\theta_1}^{\theta_2} \tfrac{1}{2} r^2 \, d\theta.$$

- To convert from cartesian form to polar form use the relationships

$$x^2 + y^2 = r^2, \qquad x = r\cos\theta, \qquad y = r\sin\theta, \qquad \tan\theta = \tfrac{y}{x}.$$

  These can be derived easily from the point (x, y) drawn with a right angled triangle to the origin.

Hyperbolic Functions

- Know that the hyperbolic trigonometric functions[3] are defined

$$\cosh x \equiv \frac{e^x + e^{-x}}{2}, \qquad \sinh x \equiv \frac{e^x - e^{-x}}{2}, \qquad \tanh x \equiv \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

[2]Otherwise $(3, 0)$, $(3, 2\pi)$, $(3, 4\pi)$, ... would all be the same point.

[3]Compare with normal trig functions $\cos x \equiv \frac{e^{ix} + e^{-ix}}{2}$, $\sin x \equiv \frac{e^{ix} - e^{-ix}}{2i}$, $\tan x \equiv \frac{\sin x}{\cos x}$.

Similarly (as you would expect) we define

$$\operatorname{sech} x \equiv \frac{1}{\cosh x}, \qquad \operatorname{cosech} x \equiv \frac{1}{\sinh x}, \qquad \coth x \equiv \frac{1}{\tanh x}.$$

- Two important relationships that drop out instantly are

$$\cosh x + \sinh x = e^x \qquad \text{and} \qquad \cosh x - \sinh x = e^{-x}.$$

- You must know (or, better yet be able to work out from the definitions) the sketches for all 6 hyperbolic curves. Also $\sinh x$ is 'one-to-one' so can be inverted without restricting the domain. However, $\cosh x$ is 'many-to-one' so a domain restriction is required ($x \geqslant 0$) to invert it.

- Differentiating the above definitions we quickly find

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x.$$

- We also find $\cosh^2 x - \sinh^2 x = 1$ and $\sinh 2x = 2\sinh x \cosh x$. To derive results like these, run back to the exponential definitions and work from one side to the other. For example to prove the latter of the two results stated, start with $2\sinh x\cosh x$:

$$\begin{aligned} 2\sinh x \cosh x &= 2\left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right) \\ &= \frac{(e^x - e^{-x})(e^x + e^{-x})}{2} \\ &= \frac{e^{2x} + 1 - 1 - e^{-2x}}{2} = \sinh 2x. \end{aligned}$$

- You need to know the logarithmic forms[4] for the inverse hyperbolic functions:

$$\begin{aligned} \sinh^{-1} x &= \ln\left(x + \sqrt{x^2+1}\right) && \text{for all } x, \\ \cosh^{-1} x &= \ln\left(x + \sqrt{x^2-1}\right) && \text{for } x \geqslant 1, \\ \tanh^{-1} x &= \tfrac{1}{2}\ln\left(\frac{1+x}{1-x}\right) && \text{for } -1 < x < 1. \end{aligned}$$

- For example: Solve $24\cosh x + 16\sinh x = 21$. Re-write the $\sinh x$ and $\cosh x$ in terms of e^x and then solve the resulting 'quadratic in disguise'.

$$\begin{aligned} 12(e^x + e^{-x}) + 8(e^x - e^{-x}) &= 21, \\ 20e^x - 21 + 4e^{-x} &= 0, \\ 20(e^x)^2 - 21(e^x) + 4 &= 0, \\ (5e^x - 4)(4e^x - 1) &= 0. \end{aligned}$$

  This then solves to $x = \ln\frac{4}{5}$ or $x = \ln\frac{1}{4}$.

- A particularly useful identity which helps in some tougher problems is $(x-\sqrt{x^2-1})(x+\sqrt{x^2-1}) \equiv 1$. So

$$\frac{1}{x - \sqrt{x^2-1}} \equiv x + \sqrt{x^2-1} \qquad \text{and} \qquad \frac{1}{x + \sqrt{x^2-1}} \equiv x - \sqrt{x^2-1}.$$

[4]Derived by considering $y = \sinh^{-1} x$, $\Rightarrow$ $\sinh y = x$, $\Rightarrow$ $e^y - \frac{1}{e^y} = 2x$, $\Rightarrow$ $(e^y)^2 - 2x(e^y) - 1 = 0$. Then solve the resulting 'quadratic in disguise' for e^y.

Differentiation & Integration

- In C3 you will have seen the wonderful trick to find the derivative of $\ln x$:

$$\begin{aligned} y &= \ln x \\ e^y &= x \\ e^y &= \frac{dx}{dy} \\ \frac{1}{e^y} &= \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{x}. \end{aligned}$$

  We can use the same trick for inverse trig functions:

$$\begin{aligned} y &= \sin^{-1} x \\ \sin y &= x \\ \cos y &= \frac{dx}{dy} \\ \frac{1}{\cos y} &= \frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}. \end{aligned}$$

  Similarly we can derive the following important results (you should do so for yourself!):

$$\begin{array}{ccc} \frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}, & \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}, \\ \frac{d}{dx}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}, & \frac{d}{dx}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}, & \frac{d}{dx}\tanh^{-1} x = \frac{1}{1-x^2}. \end{array}$$

- A glance at the formula book[5] shows that the above six derivations yield results such as $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$. However, you should not allow yourself to get tied down to the formula book. I am a firm believer that the formula booklet should act as a guide only; showing you what substitution to use. For example if we needed to find $\int \frac{5}{9+4x^2}\,dx$ we would be lost with only the formula book, because it is not in precisely the same form. However it 'looks like' the '$\tan^{-1}$' answer in the formula book, so this is the hint to use a 'tan' substitution. Here we want $4x^2 = 9\tan^2\theta$; or, more simply, $2x = 3\tan\theta$. So (deep breath!)

$$\begin{aligned} &\int \frac{5}{9+4x^2}\,dx & 2x &= 3\tan\theta \\ =&\int \frac{5}{9+9\tan^2\theta}\frac{3}{2}\sec^2\theta\,d\theta & 2\,dx &= 3\sec^2\theta\,d\theta \\ =&\frac{15}{2}\int \frac{\sec^2\theta}{9\sec^2\theta}\,d\theta \\ =&\frac{5}{6}\theta + c = \frac{5}{6}\tan^{-1}\left(\frac{2x}{3}\right) + c. \end{aligned}$$

[5] If your school has not provided you with a copy, you should ask for (demand) one. It is very useful to know what's in it. However, if you're going to an interview at a top university and you say to your interviewer "I would have to look at a formula book to answer that" then you can expect a rejection letter soon after.

- Completing the square is another useful thing to look for. Here I don't necessarily mean the strict C1 method where $9x^2+6x-15$ becomes $9(x+\frac{1}{3})^2-16$. A much more useful form for the former is $(3x+1)^2-16$, keeping everything in integers. So if asked to work out $\int \frac{7}{\sqrt{9x^2+6x-15}}\,dx$ we can re-write it as $\int \frac{7}{\sqrt{(3x+1)^2-16}}\,dx$. This looks very similar to the '$\cosh^{-1}$' differential above. Therefore we need a 'cosh' substitution. Here we want $(3x+1)^2 = 16\cosh^2 u$; or, more simply, $3x+1 = 4\cosh u$. So (here we go!)

$$\begin{aligned}
&\int \frac{7}{\sqrt{(3x+1)^2-16}}\,dx && 3x+1 = 4\cosh u\\
=&\int \frac{7}{\sqrt{16\cosh^2 u-16}}\frac{4}{3}\sinh u\,du && 3\,dx = 4\sinh u\,du\\
=&\frac{28}{3}\int \frac{\sinh u}{4\sinh u}\,du\\
=&\frac{7}{3}u + c = \frac{7}{3}\cosh^{-1}\left(\frac{3x+1}{4}\right) + c.
\end{aligned}$$

- Another useful trick is to split the numerator of a fraction in an integral into two bits of more use. For example, if faced with $\int \frac{2x+3}{x^2+4x+1}\,dx$ you can split it into $\int \frac{2x+4}{x^2+4x+1} - \frac{1}{x^2+4x+1}\,dx$, each bit of which is now more easily handled.

- A useful substitution for integrals that involve trigonometric functions is $t = \tan\left(\frac{x}{2}\right)$. This is a boon because it changes horrible integrals with trig functions into new integrals with no trig at all[6]. Given this substitution it can be shown that

$$\tan x = \frac{2t}{1-t^2}, \qquad \sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}.$$

These must be learnt! When applying this, you must also use the fact that $\frac{dx}{dt} = \frac{2}{1+t^2}$, to replace the '$dx$' at the end of the integral by '$\frac{2}{1+t^2}\,dt$'. For example to evaluate $\int \frac{\sin x}{1+\cos x}\,dx$ we find

$$\begin{aligned}
\int \frac{\sin x}{1+\cos x}\,dx &= \int \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}\,dt \qquad \text{using } t = \tan\left(\frac{x}{2}\right)\\
&= \int \frac{2t}{1+t^2}\,dt\\
&= \ln(1+t^2) + c\\
&= \ln(1+\tan^2\left(\frac{x}{2}\right)) + c.
\end{aligned}$$

The only thing to add is that if you were faced with $\int \frac{\sin 10x}{1+\cos 10x}\,dx$ the substitution would be $t = \tan 5x$. This would change the 'dx' replacement (by the chain rule) to '$\frac{1}{5(1+t^2)}\,dt$'. Therefore

$$\int \frac{\sin 10x}{1+\cos 10x}\,dx = \int \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{1}{5(1+t^2)}\,dt = \frac{1}{10}\int \frac{2t}{1+t^2}\,dt.$$

Make sure you 'get' this; it is a little subtle.

[6] I don't know about you, but I *hate* trig integrals and the sooner I can get rid of the trig bits the better.

Reduction Formulae

- Reduction formulae involve integrals which do not only involve x, but also n. In general we write
$$I_n = \int (\text{something to do with } x \text{ and } n)\, dx$$
to indicate that the integral depends on n. The aim (usually) is to find a relationship between I_n and I_{n-1} or a relationship between I_n and I_{n-2} and then use this relationship to evaluate a specific integral (I_6, say). Integration by parts tends to be the method needed to find such a relationship since the integration by parts formula[7] contains an integral on each side of the equation which may be manipulated into the desired relationship. You do occasionally need to be quite cunning![8]

- For example find $\int_0^1 x^6 e^{2x}\, dx$. Clearly[9] the hint is to let $I_n = \int_0^1 x^n e^{2x}\, dx$. By parts
$$\begin{aligned} I_n &= \int_0^1 x^n e^{2x}\, dx = \left[\frac{x^n e^{2x}}{2}\right]_0^1 - \frac{n}{2}\int_0^1 x^{n-1} e^{2x}\, dx, \\ I_n &= \frac{e^2}{2} - \frac{n}{2} I_{n-1}. \end{aligned}$$
Now we have the relationship between I_n and I_{n-1} we need some 'low' integral that we can evaluate easily: I_0 fits the bill since $I_0 = \int_0^1 e^{2x}\, dx = \frac{e^2-1}{2}$. So
$$\begin{aligned} I_6 &= \frac{e^2}{2} - \frac{6}{2} I_5 \\ &= \frac{e^2}{2} - \frac{6}{2}\left(\frac{e^2}{2} - \frac{5}{2} I_4\right) \\ &= \text{...work through for yourself...} \\ &= \frac{7e^2}{8} - \frac{45}{8}. \end{aligned}$$

- 'Snapping off' bits of trig functions often helps (i.e. writing $\sin^n x$ as either $\sin x \sin^{n-1} x$ or $\sin^2 x \sin^{n-2} x$). For example find a reduction formula for $I_n = \int \cos^n x\, dx$. Snap off a '$\cos x$' and then do parts, integrating the $\cos x$ and differentiating the $\cos^{n-1} x$.
$$\begin{aligned} I_n &= \int \cos^n x\, dx = \int \cos x \cos^{n-1} x\, dx \\ &= \sin x \cos^{n-1} x + (n-1)\int \sin^2 x \cos^{n-2} x\, dx \\ &= \sin x \cos^{n-1} x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\, dx \\ I_n &= \sin x \cos^{n-1} x + (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$
Isolating I_n we find $I_n = \frac{1}{n}\sin x \cos^{n-1} x + \frac{n-1}{n} I_{n-2}$. We can find I_0 and I_1 easily enough, which means we can evaluate $I_n = \int \cos^n x\, dx$ for any positive integer n.

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[7] $\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx$.
[8] Remember the Alloway special!
[9] Hopefully you can see why letting $I_n = \int_0^1 x^7 e^{nx}\, dx$ is a dreadful idea!

Maclaurin Series

- The Maclaurin series/expansion for a function is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \cdots = \sum_{r=0}^{\infty} \frac{f^{(r)}(0)}{r!}x^r.$$

  This is a remarkable formula; it implies that you can know a function completely over *all* values of x provided you know all the derivatives of a function at *one* value of x.

- You must know (and any good candidate ought to derive for themselves) the following standard expansions:

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots && \text{valid for all } x, \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots && \text{valid for all } x, \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots && \text{valid for all } x, \\ \ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots && \text{valid for } -1 < x < 1. \end{aligned}$$

  It is good to note that the e^x series differentiates to itself and the $\sin x$ and $\cos x$ series differentiate twice to minus themselves (as they should). Also we note that if we differentiate $\ln(1+x)$ we get $\frac{1}{1+x}$ and the general binomial expansion of this (using C4 methods) is precisely what we get by differentiating our Maclaurin expansion[10].

- You rarely (if ever) need to derive a Maclaurin series from first principles[11]. What you need to do is apply the series in the 'formula booklet' to similar situations.

- For example find the Maclaurin series for $\frac{3\cos(2x)}{1+\ln(1-4x)}$. So

$$\begin{aligned} \frac{3\cos(2x)}{1+\ln(1-4x)} &= \frac{3\left(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \ldots\right)}{1 + \left((-4x) - \frac{(-4x)^2}{2} + \frac{(-4x)^3}{3} - \ldots\right)} \\ &= \frac{3 - 6x^2 + 2x^4 - \ldots}{1 - 4x - 8x^2 + \ldots} \\ &= 3 + 12x + 66x^2 + \ldots \end{aligned}$$

  To do the last step you consider the general binomial expansion on $(1 - 4x - 8x^2 + \ldots)^{-1}$.

Series & Integrals

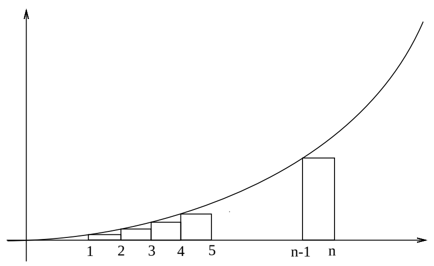
- You must be able to sandwich certain integrals between sums ($\text{sum}_1 <$ integral $< \text{sum}_2$) and sandwich certain sums between integrals ($\text{integral}_1 <$ sum $< \text{integral}_2$). The first of these is easier to formulate, but the second of these is more useful since you are currently better at integrals than sums.

- You must be abundantly clear whether you are dealing with an *increasing* or *decreasing* function and you must always draw a sketch of the relevant curve and associated rectangles to make sure you are not writing gibberish (as I have occasionally done in class). Remember a function is increasing if $\frac{dy}{dx} \geqslant 0$ for all x-values in a range. Similarly a function is decreasing if $\frac{dy}{dx} \leqslant 0$.

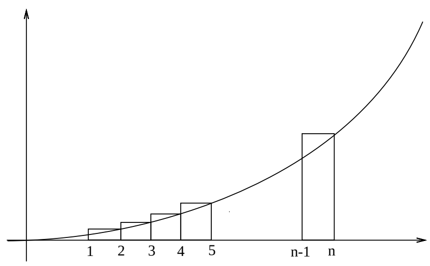
[10] $1 - x + x^2 - x^3 + \ldots$

[11] That means you McKelvie!

- For example sandwich $\int_1^n x^3\,dx$ between two sums. Firstly $y = x^3$ is an increasing function in the range stated, so if we want the sum below the integral we want the rectangles where the left height joins the curve. So $1^3 + 2^3 + \cdots + (n-1)^3 < \int_1^n x^3\,dx$. Similarly the sum above the integral is where the right height joins the curve.



and



So

$$\begin{aligned} 1^3 + 2^3 + \cdots + (n-1)^3 &< \int_1^n x^3\,dx < 2^3 + 3^3 + \cdots + n^3, \\ \sum_{i=1}^{n-1} i^3 &< \int_1^n x^3\,dx < \sum_{i=2}^{n} i^3. \end{aligned}$$

- For example sandwich $\int_1^n \frac{1}{x^2}\,dx$ between two sums. Here we have a decreasing function in the range required, so the lower limit is now given by the rectangles whose right heights join the curve. You should therefore find

$$\begin{aligned} \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^2 + \cdots + \left(\tfrac{1}{n}\right)^2 &< \int_1^n \frac{1}{x^2}\,dx < \left(\tfrac{1}{1}\right)^2 + \left(\tfrac{1}{2}\right)^2 + \cdots + \left(\tfrac{1}{n-1}\right)^2, \\ \sum_{i=2}^{n} \left(\tfrac{1}{i}\right)^2 &< \int_1^n \frac{1}{x^2}\,dx < \sum_{i=1}^{n-1} \left(\tfrac{1}{i}\right)^2. \end{aligned}$$

- Notice the way the limits on the sums seem to *flip* between increasing and decreasing functions.

- Sandwiching a sum between two integrals is a little more fiddly. For example sandwich $\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n}$ between two integrals. Clearly the function we are considering here is $y = \sqrt[3]{x}$; this is an increasing function in the range. By considering two suitable sketches we find

$$\begin{aligned} \int_0^n \sqrt[3]{x}\,dx &< \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\,dx, \\ \frac{3n^{4/3}}{4} &< \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \cdots + \sqrt[3]{n} < \frac{3(n+1)^{4/3} - 3}{4}. \end{aligned}$$

- Similarly if you want to sandwich $f(1) + f(2) + \cdots + f(n)$ between two integrals where $y = f(x)$ is a *decreasing* function for $1 < x < n$ you should find

$$\int_1^{n+1} f(x)\,dx < f(1) + f(2) + \cdots + f(n) < \int_0^n f(x)\,dx.$$

Draw a sketch to see why.

Numerical Methods

- In C3 you considered iterations of the form $x_{n+1} = F(x_n)$ which give a progression of values $x_0, x_1, x_2, \ldots x_i, \ldots x_n \ldots$ which hopefully *converge* towards a solution of the equation $x = F(x)$. The 'true value' of the solution is denoted α. We define the *error* at any point of the iteration to be the difference between the 'true value' and the value of the iteration at that point; i.e.

$$e_n = \alpha - x_n \qquad \text{or} \qquad e_i = \alpha - x_i$$

It is obviously to be hoped that e_i's get smaller as the iteration progresses.

- [Taylor series (which are not technically on the FP2 syllabus, but are needed for a full understanding of what follows) are a generalisation of Maclaurin series. Whereas Maclaurin series are 'centred around' $x = 0$ and provide increasingly good approximations to a function around the y-axis, Taylor series provide approximations to a function around any x-value you choose. The Taylor expansion around $x = a$ is

$$F(x) = F(a) + F'(a)(x-a) + \frac{F''(a)}{2!}(x-a)^2 + \frac{F'''(a)}{3!}(x-a)^3 + \cdots.$$

- When considering the iteration $x_{n+1} = F(x_n)$ we can Taylor expand $F(x_n)$ about the root α, so

$$x_{n+1} = F(x_n) = F(\alpha) + F'(\alpha)(x_n - \alpha) + \frac{F''(\alpha)}{2!}(x_n-\alpha)^2 + \frac{F'''(\alpha)}{3!}(x_n-\alpha)^3 + \cdots.$$

But note that $F(\alpha) = \alpha$ because we are solving the equation $x = F(x)$. Therefore $x_{n+1} = \alpha + F'(\alpha)(x_n - \alpha) + \frac{F''(\alpha)}{2!}(x_n-\alpha)^2 + \frac{F'''(\alpha)}{3!}(x_n-\alpha)^3 + \cdots.$]

- If $\boxed{F'(\alpha) \neq 0}$ and we are in the neighbourhood of (i.e. close to) the root we can truncate the Taylor series at the $(x_n - \alpha)$ term to obtain

$$x_{n+1} \approx \alpha + F'(\alpha)(x_n - \alpha).$$

Rearranging we find $\alpha - x_{n+1} \approx F'(\alpha)(\alpha - x_n)$ which gives $e_{n+1} \approx F'(\alpha)e_n$ so

$$\boxed{\frac{e_{n+1}}{e_n} \approx F'(\alpha) \approx \text{constant.}}$$

This shows that we require $-1 < F'(\alpha) < 1$ to get the desired *convergence* because we need the errors to get smaller as we iterate.

- If $\boxed{F'(\alpha) = 0}$ the second term in the Taylor series vanishes which means we need the next term, so

$$x_{n+1} \approx \alpha + \frac{F''(\alpha)}{2!}(x_n - \alpha)^2.$$

This rearranges to $\alpha - x_{n+1} \approx -\frac{F''(\alpha)}{2!}(\alpha - x_n)^2$ and so $e_{n+1} \approx -\frac{F''(\alpha)}{2!}e_n^2$ and therefore

$$\boxed{e_{n+1} \propto (e_n)^2;}$$

this is called *quadratic convergence* and these iterations converges quickly because if e_n is small (close to the root) then $(e_n)^2$ is much smaller (e.g. $0.01^2 = 0.0001$).

- The Newton-Raphson Method for numerical solution of equations is an ingenious method which takes the tangent to a curve at a point and uses its x-axis intercept as the next value for the iteration. For a given start value $x = x_1$ the iteration is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

  This is derived thus:

  - Start with x_n,
  - Go to the curve at this point $(x_n, f(x_n))$,
  - Construct tangent using $f'(x_n)$ as the gradient and $y - y_1 = m(x - x_1)$,
  - $y - f(x_n) = f'(x_n)(x - x_n)$,
  - Put $y = 0$ to find where tangent crosses x-axis,
  - This x value is x_{n+1}.

  Newton-Raphson converges quadratically[12] explaining its speed.

[12]The N-R iteration $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ can be thought of as $x_{n+1} = F(x_n)$ and we wish to show that $F'(\alpha) = 0$ for quadratic convergence. So differentiating $F(x)$ we find

$$\begin{aligned} F'(x) &= \frac{d}{dx}(F(x)) \\ &= \frac{d}{dx}\left(x - \frac{f(x)}{f'(x)}\right) \\ &= 1 - \left(\frac{f'(x)f'(x) - f(x)f''(x)}{[f'(x)]^2}\right) \text{ (by the quotient rule)} \\ &= \frac{f(x)f''(x)}{[f'(x)]^2} \end{aligned}$$

Putting in $x = \alpha$ we obtain $F'(\alpha) = \frac{f(\alpha)f''(\alpha)}{[f'(\alpha)]^2}$. However $f(\alpha)$ must be zero because we are solving $f(x) = 0$ and α is a root. Therefore $F'(\alpha) = 0$ when using N-R $\Rightarrow$ quadratic convergence.